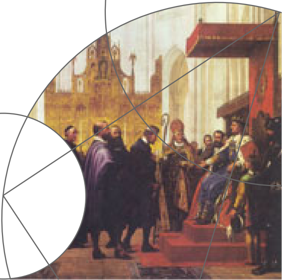
Faculty of Science

Cointegrated Oscillators

Jacob Østergaard



Agenda

1. Cointegration 101
2. Oscillators
3. Cointegrated oscillators
4. Simulation
5. Challenges
6. Current work

Cointegration 101

Cointegration 101: Integrated VAR process

We look at a multivariate Vector Auto Regressive (VAR) process $X_t \in \mathbb{R}^p$

$$X_t = \sum_{i=1}^{k} A_i X_{t-i} + \varepsilon_t,$$

where some (or all) of the variables in $X_t = (X_{1t}, \ldots, X_{pt})'$ are *integrated* of order 1 (or higher).

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For an integrated $I(1)$ process, X_t, then ΔX_t is a stationary process, which we denote $I(0)$.

Cointegration 101: VECM representation

With some algebra manipulations, we can rewrite X_t as

$$\Delta X_t = \Pi X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \varepsilon_t, \tag{1}$$

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where

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So Π holds information on stationary $I(0)$ relations of X_t.

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Let $\alpha_\perp$ be a $p \times (p - r)$ matrix, such that $\alpha'_\perp \alpha = 0$.Then with a few assumptions fulfilled, the *Granger Representation Theorem*, shows that the $p - r$ stochastic trends in the system are

$$\alpha'_\perp \sum_{i=1}^{t} \varepsilon_i$$

Cointegration 101: A quick example

Let $X_t \in \mathbb{R}^2$ and

$$\Delta X_{1t} = \alpha_1 (X_{1,t-1} - X_{2,t-1}) + \varepsilon_{1t}$$
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Since $\beta = (1, -1)'$ and $\alpha_\perp = (-\alpha_2, \alpha_1)'$ we have that

$$\beta' X_t = X_{1t} - X_{2t} = (1 + \alpha_1 - \alpha_2)(X_{1,t-1} - X_{2,t-1}) + \varepsilon_{1t} - \varepsilon_{2t}$$

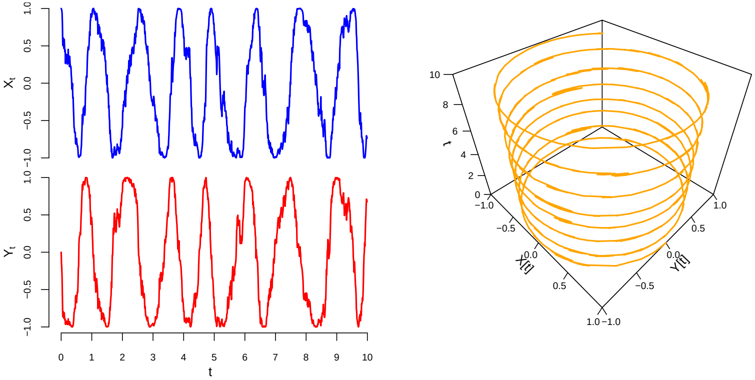
$$\alpha_\perp' X_t = -\alpha_2 X_{1t} + \alpha_1 X_{2t} = \alpha_\perp' \sum_{i=1}^{t} \varepsilon_i$$

and we have accounted for both trends in the bivariate process X_t.

Oscillators

Oscillators: A simple example

Assume a bivariate process $Z_t = (X_t, Y_t)'$, such that we observe something like this



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Et voila!

Oscillators: Multivariate version

We can generalize to a system of p oscillators, with phase-process $\phi_t \in \mathbb{R}^p$ (and $\gamma_t \in \mathbb{R}^p$).

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$$X_{it} = \gamma_{it} \cos(\phi_{it})$$
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Oscillators: Kuramoto

Now we want to introduce *synchronization* into our system. For this we need interaction between phases.

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A classical model of coupled phases is the Kuramoto model

$$d\phi_{it} = \Big(\frac{a_i}{p}\sum_{j=1}^{p}\sin(\phi_{jt}-\phi_{it}) + b_i\Big)dt + \sigma_i dW_{it}, i = 1, \dots, p.$$

This model has been studied extensively within the field of dynamical systems.

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A commonly used measure is the *mean phase coherence*

$$R_{(m,n)} = \left| \frac{1}{p} \sum_{j=1}^{p} e^{i(m\phi_{1,t_j} - n\phi_{2,t_j})} \right| \in [0,1]$$

for $m:n$ synchronization between phases ϕ_{1t} and ϕ_{2t} wrapped onto $[0,2\pi)$. Here i denotes the imaginary number.

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$R \approx 1$ implies synchronized or *phase locked* oscillators, whereas $R \approx 0$ implies unsynchronized or independent oscillators.

No specific limit defines synchronization...!

So we want to try something else.

Cointegrated oscillators

Cointegrated oscillators: The setup I

Assume a multivariate phase-process $\phi_t \in \mathbb{R}^p$, such that

$$d\phi_t = \big(f(\phi_t) + b\big)dt + \Sigma dW_t,$$

where

$$\begin{aligned} f :&\ \ \mathbb{R}^p \to \mathbb{R}^p \\ b &\in \mathbb{R}^p \\ \Sigma &= \mathrm{diag}(\sigma_1, \dots, \sigma_p), \end{aligned}$$

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The idea is to interpret this as a cointegration model.

Cointegrated oscillators: The setup II

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for $p \times r$ matrices α, β both with full column rank $r \leq p$.

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Recall that

$$\begin{aligned} X_{it} &= \gamma_t \cos(\phi_{it}) \\ Y_{it} &= \gamma_t \sin(\phi_{it}) \\ Z_{it} &= (X_{it}, Y_{it})', \end{aligned}$$

for $i = 1, \ldots, p$, and γ_t is a non-negative process.

Cointegrated oscillators: Simulating data I

Using Itô's Lemma we can derive the SDE for the data generating process (d.g.p.) $Z_t = (Z_{1t}, \dots, Z_{pt})'$, with the i'th oscillator evolving as

$$dZ_{it} = \begin{pmatrix} \frac{-\sigma_i^2}{2} & -(g(Z_t)_i + b_i) \\ g(Z_t)_i + b_i & \frac{-\sigma_i^2}{2} \end{pmatrix} Z_{it} dt + \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} Z_{it} dW_{it}^{\phi} + Z_{it} \frac{d\gamma_{it}}{\gamma_{it}},$$

where dW_{it}^{ϕ} and $d\gamma_{it}$ are uni-variate processes, and

$$g(Z_t) = f(\phi_t) = \alpha\beta'\phi_t \quad \text{is } p \times 1$$

and $g(Z_t)_i$ denotes the i'th component of $g(Z_t)$.

Cointegrated oscillators: Simulating data II

We solve for Z_t numerically, and obtain ϕ_t by

$$\phi_{it} = \text{atan2}(Y_{it}, X_{it}) + 2\pi k_{it},$$

where k_{it} is the number of rotations at time t for Z_{it} and $\text{atan2}(Y_{it}, X_{it}) \in [0, 2\pi)$.

This way we obtain the *unwrapped* phases, such that $\phi_t \in \mathbb{R}^p$.

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With our constructed d.g.p. Z_t, the phase process ϕ_t is a *continuous time* cointegration model.

Such a model can be studied with standard procedures for discrete time cointegration models, noting that unique identification of parameters can be problematic.

Simulation

Simulation: Parameters

Let $Z_t \in \mathbb{R}^6$ (3 oscillators $\Rightarrow$ 6-dimensions for the d.g.p.) and set

$$\alpha = (0.25, 0.5, 0)'$$
$$\beta = (1, -1, 0)'$$
$$b = (5, 5.1, 5)'$$
$$\sigma_1 = \sigma_2 = \sigma_3 = 1,$$

and fix $\gamma_t = (1, 1, 1)'$, so it is a constant process.

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We see that

$$\Pi = \alpha\beta' = \begin{pmatrix} 0.25 & -0.25 & 0 \\ 0.50 & -0.50 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

ie. we have two coupled oscillators and one independent, and we have that $\text{rank}(\Pi) = r_0 = 1$.

Simulation: The observed process

We use 50k simulations with timestep $dt = 0.001$, and subsample at a rate $dt = 0.01$, hence we obtain 5000 observations.

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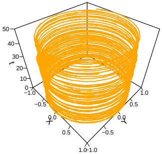
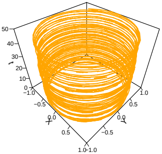
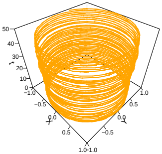
For (X_{it}, Y_{it}) vs t, they look like this...



(a) Oscillator 1



(b) Oscillator 2



(c) Oscillator 3

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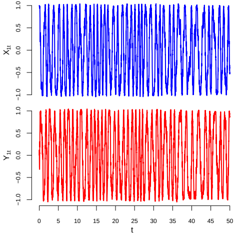
...or for X_t vs t, Y_t vs t



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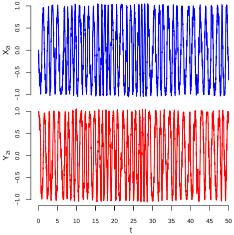
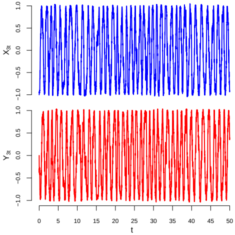


(b) Oscillator 2



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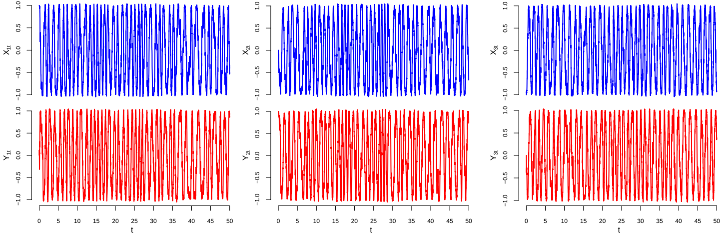
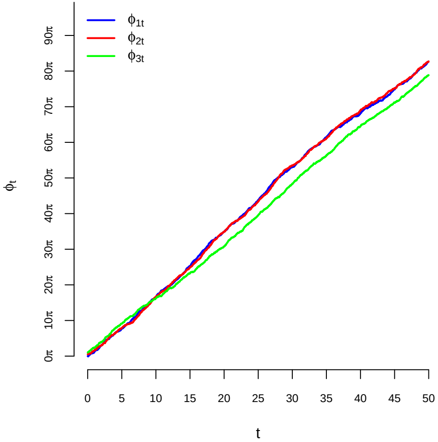


(a) Oscillator 1 (b) Oscillator 2 (c) Oscillator 3

Nothing particular to be seen here...

Simulation: The phase process

By unwrapping the atan2$(Y_{it}, X_{it}) \in [0, 2\pi)$ process we find the phase process ϕ_t



Simulation: Analysis I

A standard cointegration analysis yields

Johansen rank test (for Π)

$H(r)$	Value	10%	5%	1%
$r = 0$	85.18	28.71	31.52	37.22
$r \leq 1$	9.14	15.66	17.95	23.52
$r \leq 2$	2.62	6.50	8.18	11.65

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The test finds the true rank $r_0 = 1$ and estimates

$$\hat{\beta} = \begin{pmatrix} 1 \\ -0.94566 \\ -0.05574 \end{pmatrix}, \text{ true } \beta = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Simulation: Analysis II

Fitting a model with $r = 1$ yields the results

Parameter	True	Estimate	Std. Error	t value	p value
α_1	0.25	0.31058	0.09758	3.18274	0.00147
α_2	0.50	0.79601	0.09830	8.09735	$<$ 1e-15
α_3	0.00	0.12363	0.09812	1.25999	0.20773
b_1	5.00	5.11907	0.18659	27.43507	$<$ 1e-15
b_2	5.10	4.93054	0.18797	26.23018	$<$ 1e-15
b_3	5.00	4.81664	0.18761	25.67311	$<$ 1e-15

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We roughly recover our initial parameters, but especially the significance of the relevant parameters.

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b_1	5.00	5.11907	0.18659	27.43507	$<$ 1e-15
b_2	5.10	4.93054	0.18797	26.23018	$<$ 1e-15
b_3	5.00	4.81664	0.18761	25.67311	$<$ 1e-15

We roughly recover our initial parameters, but especially the significance of the relevant parameters.

Recall that we have an identification issue with a continuous cointegration model.

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Unfortunately we are quite far from the land of milk and honey...

Challenges

Challenges: Kuramoto is nonlinear...

Recall the Kuramoto model

$$d\phi_{it} = \left(a_i \sum_{j=1}^{p} \sin(\phi_{jt} - \phi_{it}) + b_i\right) dt + \sigma_i dW_{it}.$$

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We can write up a 3-dimensional version

$$d\phi_t = \big(f(\phi_t) + b\big) dt + \Sigma dW_t,$$

where

$$f(\phi_t) = \begin{pmatrix} a_1 & a_1 & 0 \\ -a_2 & 0 & a_2 \\ 0 & -a_3 & -a_3 \end{pmatrix} \begin{pmatrix} \sin(\phi_{2t} - \phi_{1t}) \\ \sin(\phi_{3t} - \phi_{1t}) \\ \sin(\phi_{3t} - \phi_{2t}) \end{pmatrix}.$$

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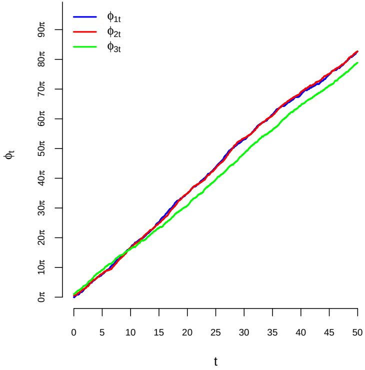

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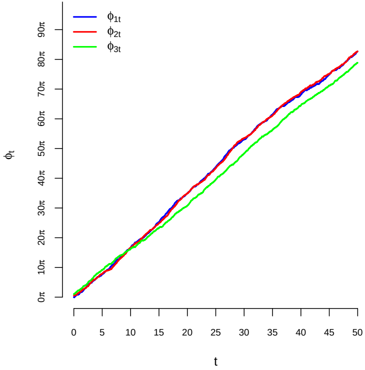
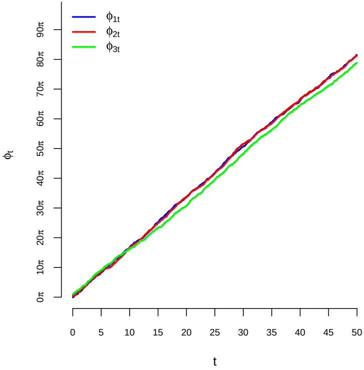


... and compare with Kuramoto



Challenges: From linear to nonlinear

Our current cointegration formulation is

$$f_{linear}(\phi_t) = \alpha\beta'\phi_t = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 & \alpha_1\beta_3 \\ \alpha_2\beta_1 & \alpha_2\beta_2 & \alpha_2\beta_3 \\ \alpha_3\beta_1 & \alpha_3\beta_2 & \alpha_3\beta_3 \end{pmatrix} \begin{pmatrix} \phi_{1t} \\ \phi_{2t} \\ \phi_{3t} \end{pmatrix}$$

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Not very surprising...

We need something like $\alpha\beta' f(\phi_t)$ for this problem, where $|f(\phi_t)_i| \leq 1$, for $i = 1, 2, 3$.

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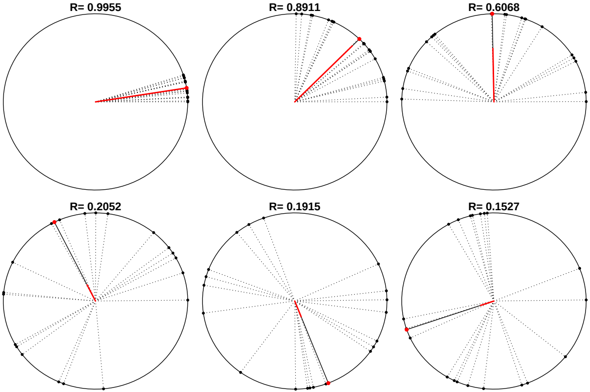

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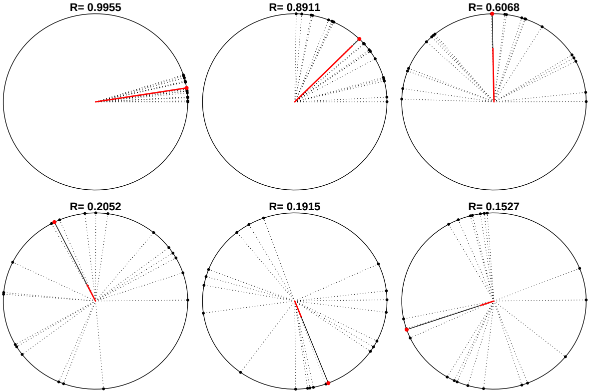

So we might need a different interpretation of *synchronization* in a cointegration context...

Current work

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The general question is:

How to best approximate Kuramoto?

Use a piecewise (non-)linear approximation to Kuramoto, or something completely new, like $\alpha\beta' f(\phi_t)$?

Current work: Tasks

- Construct a (possibly nonlinear) cointegration approximation/representation of the Kuramoto model.
- Define synchronization in a cointegration context.
- Analyze model asymptotics.
- Consider a γ_t process to replicate other well known oscillating systems.
- Implement bootstrapping for critical values for the rank test, for a nonlinear model.
- Threshold cointegration $\Rightarrow$ switch synchronization on/off?
- Analysis of nephron data: estimate internal organ structure.

?

Thank you!

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