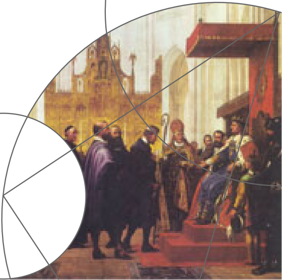
Faculty of Science

Cointegrated Oscillating Dynamical Systems

Jacob Østergaard



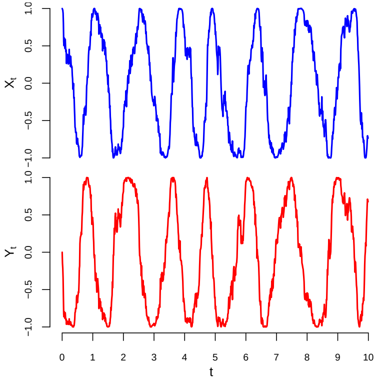
DSIN Meeting, April 19, 2016

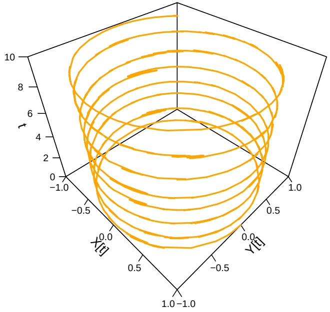
Agenda

Oscillating Systems

Oscillating Systems: A simple example

Assume a bivariate process $Z_t = (X_t, Y_t)'$, such that we observe something like this





Oscillating Systems: The Phase Process

We define an *oscillator* through a *phase-process*:

Define $\phi_t \in \mathbb{R}$ by the SDE

$$d\phi_t = bdt + \sigma dW_t,$$

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for some, possibly stochastic, non-negative process γ_t.

Note that:

ϕ_t is the angle (phase) and
γ_t is the radius.

Oscillating Systems: Multivariate version

We generalize to a system of p oscillators, with phase-process $\phi_t \in \mathbb{R}^p$ (and $\gamma_t \in \mathbb{R}^p$).

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$$X_{it} = \gamma_{it} \cos(\phi_{it})$$
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for $i = 1, \ldots, p$.

Oscillating Systems: Synchronization

The Kuramoto model: a classical model of coupled phases.

$$d\phi_{it} = \big(\frac{a_i}{p}\sum_{j=1}^{p}\sin(\phi_{jt} - \phi_{it}) + b_i\big)dt + \sigma_i dW_{it}, i = 1, \ldots, p.$$

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We define $\phi_t \in \mathbb{R}^p$ as

$$d\phi_t = \big(f(\phi_t) + \mu_t\big)dt + \Sigma dW_t,$$

with $f(\phi_t) : \mathbb{R}^p \to \mathbb{R}^p$.

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Oscillating Systems: Measuring synchronization

The *mean phase coherence* is measure of synchronization:

$$R_{(m,n)} = \left| \frac{1}{T} \sum_{t=1}^{T} e^{i(m\phi_{\mathbf{1},t} - n\phi_{\mathbf{2},t})} \right| \in [0,1]$$

measure $m:n$ synchronization between phases ϕ_{1t} and ϕ_{2t} wrapped onto $[0, 2\pi)$. Here i denotes the imaginary number.

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$R \approx 1 \Rightarrow$ synchronized or *phase locked* oscillators,
$R \approx 0 \Rightarrow$ unsynchronized/independent oscillators.

- No specific limit defines synchronization.
- Symmetric $\Rightarrow$ cannot detect uni-directional coupling.

Cointegrated phases

Cointegrated phases: The setup

Recall the multivariate phase-process $\phi_t \in \mathbb{R}^p$, such that

$$d\phi_t = \big(f(\phi_t) + \mu\big)dt + \Sigma dW_t,$$

where

$$\begin{aligned} &f: \ \mathbb{R}^p \to \mathbb{R}^p \\ &\mu \in \mathbb{R}^p \\ &\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_p), \end{aligned}$$

and $f(\phi_t)$ is linear.

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We analyze ϕ_t using *cointegration* theory.

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Let

$$f(\phi_t) = \Pi\phi_t = \alpha\beta'\phi_t,$$

for $\alpha, \beta \in \mathbb{R}^{p\times r}$ with full column rank $r < p$.

Cointegrated phases: DGP

With

$$Z_{it} = (X_{it}, Y_{it})' = (\gamma_t \cos(\phi_{it}), \gamma_t \sin(\phi_{it}))',$$

using Itô's Lemma we find that the i'th oscillator evolve as

$$dZ_{it} = \begin{pmatrix} \frac{-\sigma_i^2}{2} & -(g(Z_t)_i + b_i) \\ g(Z_t)_i + b_i & \frac{-\sigma_i^2}{2} \end{pmatrix} Z_{it} dt + \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} Z_{it} dW_{it} + Z_{it} \frac{d\gamma_{it}}{\gamma_{it}},$$

where dW_{it} and $d\gamma_{it}$ are uni-variate processes, and

$$g(Z_t) = f(\phi_t) = \alpha\beta'\phi_t \quad \text{is } p \times 1$$

such that $g(Z_t)_i$ denotes the i'th component of $g(Z_t)$.

Cointegrated phases: Simulating systems

Solving for Z_t numerically, then ϕ_t is obtained by

$$\phi_{it} = \text{atan2}(Y_{it}, X_{it}) + 2\pi k_{it},$$

with k_{it} the number of rotations at time t for Z_{it}, and $\text{atan2}(Y_{it}, X_{it}) \in [0, 2\pi)$.

This way we get the *unwrapped* phases $\phi_t \in \mathbb{R}^p$.

Simulation

Simulation: Overall Settings

Shared parameters:

$$\begin{aligned} p &= 3 \\ \mu_t &= (6, 5, 5)' \ \ \forall t \\ \sigma_1 &= \sigma_2 = \sigma_3 = 1 \\ \gamma_t &= (1, 1, 1)' \ \ \forall t \\ z_0 &= (1, 0, 0, 1, -1, 0)' \end{aligned}$$

Simulate 100k steps with $\tilde{\Delta} t = 0.001$ using Euler-Maruyama, then subsample every 100th to obtain 1000 observations with timestep $\Delta t = 0.1$, i.e. $t \in [0, 100]$.

Simulation: Systems

Four different systems

$$\Pi_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Pi_1 = \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Pi_2 = \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0.5 & -0.5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Pi_3 = \begin{pmatrix} -0.5 & 0.25 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.25 & 0.25 & -0.5 \end{pmatrix}$$

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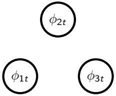
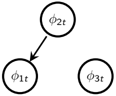
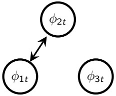
Π_0 :



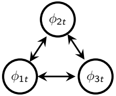
Π_1 :



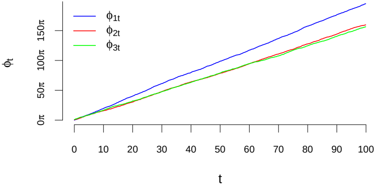
Π_2 :



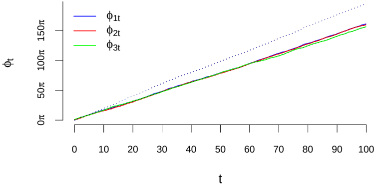
Π_3 :



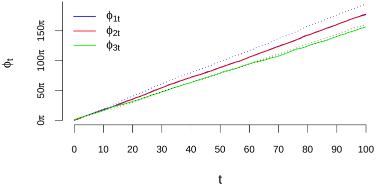
Simulation: Phases for the 4 systems



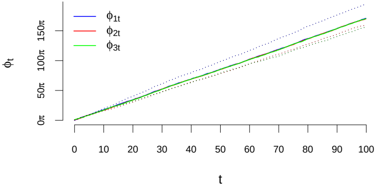
(a) Π_0 model.



(b) Π_1 model.



(c) Π_2 model.



(d) Π_3 model.

Analysis

Analysis: Rank tests for rank(Π)

Johansen rank tests and mean phase coherence measures.

Model	H_r	Test values	p-value	Mean Phase Coherence	
Π_0	$r = 0$	14.94	0.751	$R(\phi_{1t}, \phi_{2t})$	0.055
	$r \leq 1$	6.73	0.519	$R(\phi_{1t}, \phi_{3t})$	0.083
	$r \leq 2$	0.17	0.635	$R(\phi_{2t}, \phi_{3t})$	0.098
Π_1	$r = 0$	52.50	0.000	$R(\phi_{1t}, \phi_{2t})$	0.415
	$r \leq 1$	5.61	0.489	$R(\phi_{1t}, \phi_{3t})$	0.199
	$r \leq 2$	0.78	0.306	$R(\phi_{2t}, \phi_{3t})$	0.098
Π_2	$r = 0$	64.78	0.000	$R(\phi_{1t}, \phi_{2t})$	0.646
	$r \leq 1$	6.57	0.305	$R(\phi_{1t}, \phi_{3t})$	0.084
	$r \leq 2$	0.00	0.983	$R(\phi_{2t}, \phi_{3t})$	0.135
Π_3	$r = 0$	77.39	0.000	$R(\phi_{1t}, \phi_{2t})$	0.560
	$r \leq 1$	33.24	0.000	$R(\phi_{1t}, \phi_{3t})$	0.424
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Analysis: Π_1 model

Recall

$$\Pi_1 = \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has $\text{rank}(\Pi_1) = 1 \Rightarrow$ we can use ($\Pi_1 = \alpha\beta'$):

$$\alpha = (-0.5, 0, 0)'$$
$$\beta = (1, -1, 0)',$$

both with full column rank $r = 1$.

We start by fitting the model with $r = 1$, for $\alpha, \beta \in \mathbb{R}^{3\times 1}$

Analysis: Π_1 model

Fitted model Π_1 with unrestricted α, β:

Para-meter	True value	Unrestricted α, β Estimate	Std. Error	p value
α_1	-0.5	-0.471	0.072	< 0.001
α_2	0	0.074	0.075	0.329
α_3	0	-0.121	0.077	0.117
β_1	1	1		
β_2	-1	-1.028		
β_3	0	0.031		
μ_1	6	6.321	0.214	< 0.001
μ_2	5	4.810	0.224	< 0.001
μ_3	5	5.209	0.230	< 0.001

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$$H_{\alpha,\beta}: \quad \alpha = A\psi, \text{ with } A = (1,0,0)'$$
$$\beta = B\xi, \text{ with } B = (1,-1,0)'$$

Analysis: Π_1 model

Fitted model Π_1 under $H_{\alpha,\beta}$ (p-value is 0.365):

Parameter	True value	Restricted α, β Estimate	Std. Error	p value
α_1	-0.5	-0.469	0.072	< 0.001
α_2	0	0		
α_3	0	0		
β_1	1	1		
β_2	-1	-1		
β_3	0	0		
μ_1	6	6.066	0.180	< 0.001
μ_2	5	5.006	0.188	< 0.001
μ_3	5	4.886	0.193	< 0.001

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μ_2	5	5.006	0.188	< 0.001
μ_3	5	4.886	0.193	< 0.001

Conclusion: We recover the correct uni-directional coupling structure.

Analysis: Π_2 model

Recall

$$\Pi_2 = \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0.5 & -0.5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has $\text{rank}(\Pi_2) = 1 \Rightarrow$ we can use ($\Pi_2 = \alpha\beta'$):

$$\alpha = (-0.5, 0.5, 0)'$$
$$\beta = (1, -1, 0)'.$$

both with full column rank $r = 1$.

We start by fitting the model with $r = 1$, for $\alpha, \beta \in \mathbb{R}^{3\times 1}$

Analysis: Π_2 model

Fitted model Π_2 with unrestricted α, β:

Parameter	True value	Unrestricted α, β Estimate	Std. Error	p value
α_1	-0.5	-0.437	0.103	< 0.001
α_2	0.5	0.593	0.105	< 0.001
α_3	0	-0.190	0.110	0.083
β_1	1	1		
β_2	-1	-0.989		
β_3	0	-0.012		
μ_1	6	6.160	0.164	< 0.001
μ_2	5	4.777	0.167	< 0.001
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$$H_{\alpha,\beta} : \alpha = A\psi, \text{ with } A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \beta = B\xi, \text{ with } B = (1, -1, 0)'$$

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Fitted model Π_2 under $H_{\alpha,\beta}$ (p-value is 0.164):

Para-meter	True value	Unrestricted α, β Estimate	Std. Error	p value
α_1	-0.5	-0.420	0.101	< 0.001
α_2	0.5	0.574	0.103	< 0.001
α_3	0	0		
β_1	1	1		
β_2	-1	-1		
β_3	0	0		
μ_1	6	6.045	0.144	< 0.001
μ_2	5	4.929	0.148	< 0.001
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Analysis: Π_2 model

Fitted model Π_2 under $H^*_{\alpha,\beta}$ (p-value is 0.187):

Parameter	True value	Unrestricted α, β Estimate	Std. Error	p value
α_1	-0.5	-0.497	0.101	< 0.001
α_2	0.5	0.497	0.103	< 0.001
α_3	0	0		
β_1	1	1		
β_2	-1	-1		
β_3	0	0		
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Conclusion: We recover the correct bi-directional coupling structure.

Analysis: Π_3 model

Recall

$$\Pi_3 = \begin{pmatrix} -0.5 & 0.25 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.25 & 0.25 & -0.5 \end{pmatrix}$$

has $\text{rank}(\Pi_3) = 2 \Rightarrow$ we can use ($\Pi_3 = \alpha\beta'$):

$$\alpha = \begin{pmatrix} -0.50 & 0.25 \\ 0.25 & -0.50 \\ 0.25 & 0.25 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

both with full column rank $r = 2$.

We start by fitting the model with $r = 2$, for $\alpha, \beta \in \mathbb{R}^{3\times 2}$

Analysis: Π_3 model

Fitted model Π_3 with unrestricted α, β:

Para-meter	True value	Unrestricted β Estimate	Std. Error	p value
α_{11}	-0.50	-0.248	0.074	0.001
α_{21}	0.25	0.374	0.064	< 0.001
α_{31}	0.25	0.184	0.076	0.015
α_{12}	0.25	0.226	0.065	0.001
α_{22}	-0.50	-0.033	0.078	0.673
α_{32}	0.25	-0.301	0.067	< 0.001
β_{11}	1	1		
β_{21}	0	-1.409		
β_{31}	-1	0.410		
β_{12}	0	-0.865		
β_{22}	1	0.344		
β_{32}	-1	1.209		
μ_1	6	6.196	0.194	< 0.001
μ_2	5	4.712	0.199	< 0.001
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Doesn't look so good...

Analysis: Π_3 model

Look at the estimated $\hat{\Pi}$:

$$\hat{\Pi} = \hat{\alpha}\hat{\beta}' = \begin{pmatrix} -0.444 & 0.272 & 0.171 \\ 0.346 & -0.539 & 0.193 \\ 0.076 & 0.363 & -0.439 \end{pmatrix}$$

vs

$$\Pi_3 = \begin{pmatrix} -0.5 & 0.25 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.25 & 0.25 & -0.5 \end{pmatrix}$$

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- Both $\sum_j^3 \hat{\Pi}_{ij} \approx 0$ and $\sum_i^3 \hat{\Pi}_{ij} \approx 0$.

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- Both $\sum_j^3 \hat{\Pi}_{ij} \approx 0$ and $\sum_i^3 \hat{\Pi}_{ij} \approx 0$.
- Identification of α, β can be problematic...

Analysis: Π_3 model

How does the true structure fit the data?

$$H_{\alpha,\beta} : \alpha = A\psi, \text{ with } A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \\ 1 & 1 \end{pmatrix}$$

$$\beta = B\xi, \text{ with } B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}$$

Analysis: Π_3 model

Fitted model Π_3 under $H_{\alpha,\beta}$ (p-value is 0.254):

Parameter	True value	Unrestricted β Estimate	Std. Error	p value
α_{11}	-0.50	-0.159	0.065	0.014
α_{21}	0.25	0.337	0.047	< 0.001
α_{31}	0.25	-0.196	0.066	0.003
α_{12}	0.25	0.113	0.048	0.020
α_{22}	-0.50	-0.055	0.068	0.423
α_{32}	0.25	-0.228	0.049	< 0.001
β_{11}	1	1		
β_{21}	0	-1.607		
β_{31}	-1	0.607		
β_{12}	0	-1.364		
β_{22}	1	-0.181		
β_{32}	-1	1.545		
μ_1	6	5.836	0.165	< 0.001
μ_2	5	4.755	0.169	< 0.001
μ_2	5	5.085	0.173	< 0.001

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Analysis: Π_3 model

The estimated $\hat{\Pi}$:

$$\hat{\Pi}^{H_{\alpha,\beta}} = \begin{pmatrix} -0.313 & 0.236 & 0.077 \\ 0.412 & -0.532 & 0.120 \\ 0.115 & 0.357 & -0.472 \end{pmatrix}$$

vs

$$\Pi_3 = \begin{pmatrix} -0.5 & 0.25 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.25 & 0.25 & -0.5 \end{pmatrix}$$

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$$\Pi_3 = \begin{pmatrix} -0.5 & 0.25 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.25 & 0.25 & -0.5 \end{pmatrix}$$

Row sums more or less zero, column sums are more off...

Analysis: Conclusions

Cointegration analysis findings:

Model	Description	Conclusion
Π_0	Independent	Independent oscillators.
Π_1	Uni-directional coupling	Uni-directional coupling.
Π_2	Bi-directional coupling	Bi-directional coupling, equal coupling strength.
Π_3	Fully coupled	Unclear*.

*: We cannot identify the correct structure from the model fit, but the data does admit a restriction to the true proportions of the parameter matrices.

Challenges

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- Learn to interpret cointegration results in terms of interacting oscillators.
- Derive a non-linear cointegration mechanism to model Kuramoto.
- Derive a framework with non-linear deterministic trends for the model.
- Analyze real data!

?

Thank you!

Jacob Østergaard

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